can usually be gained by looking at a discretization process from different points of view, and this is exactly what happens in this case.

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Constrained Motions of Systems with Elastic Members

Shlomo Djerassi*
RAFAEL, Ministry of Defense, 31021 Haifa, Israel

Introduction

ANALYSES of systems with elastic members usually start with the generation of motion equations.[1–3] Authors use a variety of procedures to generate these equations. Common to procedures based on Lagrange's formulation are the time-consuming steps such as generation of kinetic energy expressions for generic particles of the elastic members and integration of these expressions over the entire volume of the bodies.

It is the contention of this Note that motion equations of systems with elastic members can be obtained more expeditiously if use is made of motion equations for constrained systems. This contention is discussed after review of Kane's equations for constrained systems, chosen here to be the working tool.

Accordingly, let S be a simple nonholonomic system of ν particles $P_i (i = 1, \ldots, \nu)$ of mass m_i possessing $\bar{n}$ generalized coordinates $q_1, \ldots, q_{\bar{n}}$ and n (where $n \leq \bar{n}$) generalized speeds $u_1, \ldots, u_n$ in N, a Newtonian reference frame. Suppose that the motion of S in N is defined as unconstrained and is governed by n dynamical equations, namely,

$$F_r + F_r^* = 0 \qquad (r = 1, \ldots, n) \tag{1}$$

where F_r and F_r^* are the rth generalized active force and the rth generalized inertia force for S, respectively. Moreover, suppose that m constraints of the form

$$u_k = \sum_{r=1}^{p} C_{kr} u_r + D_k \qquad (k = p + 1, \ldots, n) \tag{2}$$

are imposed on the motion of S, where

$$p \triangleq n - m \tag{3}$$

and C_{kr} and D_k are functions of $q_1, \ldots, q_{\bar{n}}$ and time t. Then the motion of S in N is defined as constrained and is governed by p dynamical equations, namely,

$$F_r + F_r^* + \sum_{k=p+1}^{n} C_{kr}\left(F_k + F_k^*\right) = 0 \qquad (r = 1, \ldots, p) \tag{4}$$

These equations were first presented by Wampler et al.[4] in a matrix form.

Suppose that the members comprising the system in question are temporarily regarded as undergoing an unconstrained motion and that motion equations represented by Eqs. (1) are written for each member. Moreover, suppose that the kinematical constraint equations are formulated and cast in the form of Eqs. (2). Then substitutions in Eqs. (4) lead to the requisite motion equations. This procedure underlies works by Buffinton and Kane[5] and by Djerassi and Kane.[6] Using the elastic properties of the unconstrained members in the generation of Eqs. (1), these authors assumed that Eqs. (4)—where expressions from Eqs. (1) are used—automatically reproduce the elastic properties of the constrained system. A similar assumption was made by Thomson,[7] discussing the elastic properties of an elastic beam on three supports. The validity of this assumption cannot be proved rigorously; however, it can be illustrated, a task undertaken here in connection with two examples. These examples demonstrate the benefits associated with the exploitation of this assumption.

Cantilever Beam with a Massive Endpoint

Consider the system S comprising an elastic cantilever beam B and a particle P of mass m attached to B at its endpoint E. Suppose that B and P are temporarily regarded as undergoing an unconstrained motion. Then equations of motion of B and P can be generated independently. Accordingly, B is assumed to behave as a uniform Euler–Bernoulli beam, whose motion is governed by the equations

$$-M\dot{u}_i - EJL\lambda_i^4 q_i = 0 \qquad (i = 1, \ldots, \mu) \tag{5}$$

$$\dot{q}_i = u_i \qquad (i = 1, \ldots, \mu) \tag{6}$$

where EJ, L, and M are, respectively, the bending rigidity, the length, and the mass of the beam; and q_i, u_i, λ_i, and μ are, respectively, the ith generalized coordinate, the rth generalized speed, the ith eigenvalue of the equation

$$1 + \cos\lambda_i L \cdot \cosh\lambda_i L = 0 \tag{7}$$

and the number of modes used to describe the elastic deflection $y(x, t)$ of points of the beam. The latter can be expressed as

$$y(x, t) = \sum_{i=1}^{\mu} \phi_i(x) q_i(t) \tag{8}$$

where $\phi_i(x)$, the ith modal function, is given by

$$\phi_i(x) = -\cos\lambda_i x + \cosh\lambda_i x + k_i(\sin\lambda_i x - \sinh\lambda_i x)$$
$$k_i = \frac{\cos\lambda_i L + \cosh\lambda_i L}{\sin\lambda_i L + \sinh\lambda_i L} \tag{9}$$

if the boundary conditions are

$$y(0, t) = y'(0, t) = y''(L, t) = y'''(L, t) = 0 \tag{10}$$

In connection with the motion of P, $u_{\mu+1}$ is defined so that the velocity of P in N is expressed as ${}^N\boldsymbol{v}^P = u_{\mu+1}\boldsymbol{n}$. Then the dynamical equations governing the motion of S in N are Eqs. (5), and, in addition

$$-m\dot{u}_{\mu+1} = 0 \tag{11}$$

Consider a constrained motion of S with P attached to E. Then ${}^N\boldsymbol{v}^P = {}^N\boldsymbol{v}^E$, and because ${}^N\boldsymbol{v}^E = \dot{y}(L, t)\boldsymbol{n}$, it follows that $u_{\mu+1}(t) = \dot{y}(L, t)$, or

$$u_{\mu+1}(t) = \phi_1(L)u_1(t) + \cdots + \phi_\mu(L)u_\mu(t) \tag{12}$$

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*Research Fellow, P.O. Box 2250.

This equation is cast in the form of Eq. (2), and hence $C_{\mu+1,i} = \phi_i(L)(i = 1, \ldots, \mu)$. Substitutions from Eqs. (5) and (11) in Eqs. (4), together with $C_{\mu+1,i}(i = 1, \ldots, \mu)$, result in the following equations:

$$-M\dot{u}_i - EJL\lambda_i^4 q_i + \phi_i(L)(-m\dot{u}_{\mu+1}) = 0 \qquad (i = 1, \ldots, \mu) \tag{13}$$

where $\lambda_i(i = 1, \ldots, \mu)$ are the solutions of Eqs. (7) and for $i = 1, \ldots, \mu$, $\phi_i(x)$ is given by Eq. (9).

Reconsidering next the beam with the mass attached to its endpoint, one can show that its motion is governed by the following equations:

$$-M\dot{\bar{u}}_i - EJL\bar{\lambda}_i^4 \bar{q}_i = 0 \qquad (i = 1, \ldots, \mu) \tag{14}$$

where $\bar{q}_i(t)$ is the ith generalized coordinate and $\dot{\bar{q}}_i = \bar{u}_i(i = 1, \ldots, \mu)$. Moreover, $\bar{y}(x, t)$, denoting the elastic deflection of points of the beam, and $\bar{\phi}_i(x)$, the modal functions, are given by expressions similar to Eqs. (8) and (9). The boundary conditions, however, are

$$\bar{y}(0, t) = \bar{y}'(0, t) = \bar{y}''(L, t) = 0 \qquad EJ\bar{y}'''(L, t) = m\ddot{\bar{y}}(L, t) \tag{15}$$

so that $\bar{\lambda}_i$ is the ith solution of the following equation:

$$1 + \cos\bar{\lambda}_i L \cdot \cosh\bar{\lambda}_i L = (m/M)(\bar{\lambda}_i L)(\sin\bar{\lambda}_i L \cdot \cosh\bar{\lambda}_i L - \cos\bar{\lambda}_i L \cdot \sinh\bar{\lambda}_i L) \tag{16}$$

Thus, Eqs. (13) and (7) on the one hand and Eqs. (14) and (16) on the other hand govern the motion of S, and hence they should be equivalent. This can be demonstrated by comparison of numerical solutions of the two sets of equations, as shown in Fig. 1 for $\bar{q}_i(0) = 4$ $(i = 1, \ldots, 6)$ $\bar{u}_i(0) = 0$ $(i = 1, \ldots, 6)$, $m = M = 0.468$ kg, $EJ = 4.2$ N-m^2, $L = 1$ m and for $q_i(0)$ and $u_i(0)(i = 1, \ldots, 6)$ evaluated with the aid of the relations $\bar{y}(x, 0) = y(x, 0)$ and $\dot{\bar{y}}(x, 0) = \dot{y}(x, 0)$. Then $y(L, t)$, $y(0.52L, t)$, $\bar{y}(L, t)$ and $\bar{y}(0.52L, t)$ are as in Fig. 1a for $\mu = 6$ and as in Fig. 1b for $\mu = 9$. These results leads to the following conclusion, namely, that Eqs. (13), or in general Eqs. (4) using data associated with modal analysis of the unconstrained system [represented here by Eqs. (7) and (9)], reconstruct the properties of the constrained system and that the higher number of modes the better is the agreement between the two approaches.

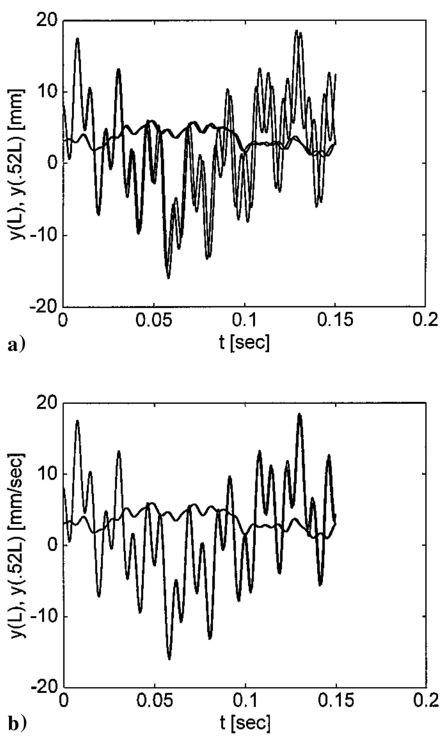

Fig. 1 Elastic deflection at E and D: direct approach and new approach with six modes (upper figure) and with nine modes.

The present method cannot lead to dependable predictions of the shearing force in cross sections of B in the constrained phase. For instance, the shearing force at E is given by $m\ddot{\bar{y}}(L, t)$ [see Eqs. (15)], whereas $EJy'''(L, t)$, the shearing force associated with Eq. (13), vanishes, in accordance with Eqs. (10). Nevertheless, methods are available for the identification of such forces.[8]

Space Robot with a Captured Satellite

Figure 2 shows a planar version of a space robot, comprising a rigid base B, two flexible arms C and D, an end effector E, and a satellite P. Suppose each of these bodies is temporarily regarded as undergoing an unconstrained motion. Moreover, suppose that points $\hat{B}$, $\bar{C}$, $\hat{C}$, $\bar{D}$, $\hat{D}$, $\bar{E}$, $\hat{E}$, and $\bar{P}$ are introduced as shown in Fig. 2; that B^*, E^*, and P^* represent the mass centers of B, E, and P, respectively; and that reference frames C' and D' are introduced, coinciding with C and D, respectively, when undeformed. Let $\boldsymbol{b}_i$, $\boldsymbol{c}_i$, $\boldsymbol{d}_i$, $\boldsymbol{e}_i$, and $\boldsymbol{p}_i(i = 1, 2, 3)$ be triads of dextral, mutually perpendicular unit vectors fixed in B, E, C', D', and P, respectively, and let $\hat{b}_i$, $\bar{e}_i$, $\hat{e}_i$, and $\bar{p}_i(i = 1, 2)$ be constants. Assume that 21 generalized speeds $u_1, \ldots, u_{21}$ and 21 generalized coordinates $q_1, \ldots, q_{21}$ are used to describe the motion and configuration of S in N, respectively, and let $\dot{q}_r = u_r(r = 1, \ldots, 21)$. Define these variables so that the angular velocities of B, C', D', E, and P, and the velocities of B^*, $\bar{C}$, $\bar{D}$, E^*, and P^* are given by

$$
\begin{aligned}
{}^N\omega^B &= u_3\boldsymbol{n}_3 & {}^N\boldsymbol{v}^{B^*} &= u_1\boldsymbol{n}_1 + u_2\boldsymbol{n}_2 \\
{}^N\omega^{C'} &= u_4\boldsymbol{n}_3 & {}^N\boldsymbol{v}^{\bar{C}} &= u_{13}\boldsymbol{n}_1 + u_{14}\boldsymbol{n}_2 \\
{}^N\omega^{D'} &= u_5\boldsymbol{n}_3 & {}^N\boldsymbol{v}^{\bar{D}} &= u_{15}\boldsymbol{n}_1 + u_{16}\boldsymbol{n}_2 \\
{}^N\omega^E &= u_6\boldsymbol{n}_3 & {}^N\boldsymbol{v}^{E^*} &= u_{17}\boldsymbol{n}_1 + u_{18}\boldsymbol{n}_2 \\
{}^N\omega^P &= u_{21}\boldsymbol{n}_3 & {}^N\boldsymbol{v}^{P^*} &= u_{19}\boldsymbol{n}_1 + u_{20}\boldsymbol{n}_2
\end{aligned} \tag{17}
$$

and the elastic deflections of points of C and D are given by

$$y^C(x, t) = \sum_{r=7}^{9} \phi_{r-6}(x)q_r(t) \qquad y^D(x, t) = \sum_{r=10}^{12} \phi_{r-9}(x)q_r(t) \tag{18}$$

Let O be a point fixed in N; let $q_1 \hat{=} \cos^{-1}(\boldsymbol{b}_1 \cdot \boldsymbol{n}_1)$ be the orientation angle of B in N; let $q_i \hat{=} \boldsymbol{p}^{O/B^*} \cdot \boldsymbol{n}_i(i = 1, 2)$ be measure numbers of the position vector $\boldsymbol{p}^{O/B^*}$ from O to B^*; and let q_4 q_{13} and q_{14}, q_5 q_{15} and q_{16}, q_6 q_{17} and q_{18}, and q_{21} q_{19} and q_{20} play similar roles in connection with C', D', E, and P and $\bar{C}$, $\bar{D}$, E^*, and P^*, respectively. Finally, express the position vectors from B^*, E^*, and P^* to $\hat{B}$, $\bar{E}$, and $\hat{E}$, and $\bar{P}$, respectively, as $\boldsymbol{p}^{B^*/\hat{B}} = \hat{b}_1\boldsymbol{b}_1 + \hat{b}_2\boldsymbol{b}_2$, $\boldsymbol{p}^{E^*/\bar{E}} = \bar{e}_1\boldsymbol{e}_1 + \bar{e}_2\boldsymbol{e}_2$, $\boldsymbol{p}^{E^*/\hat{E}} = \hat{e}_1\boldsymbol{e}_1 + \hat{e}_2\boldsymbol{e}_2$, and $\boldsymbol{p}^{P^*/\bar{P}} = \bar{p}_1\boldsymbol{p}_1 + \bar{p}_2\boldsymbol{p}_2$. Using these expressions, one can show that the motion of B in N is governed by the equations

$$-m_B\dot{u}_1 = 0 \qquad -m_B\dot{u}_2 = 0 \qquad -J_B\dot{u}_3 = 0 \tag{19}$$

and that similar equations govern the motion of E (with u_{17}, u_{18}, and u_6 replacing u_1, u_2, and u_3, respectively) and of P (with u_{19},

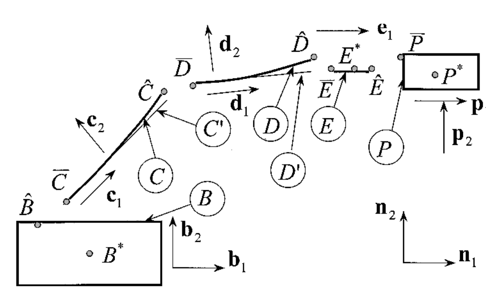

Fig. 2 Two-link robotic arm and a satellite.

u_{20}, and u_{21} replacing u_1, u_2, and u_3, respectively) if linearized in all of the motion and configuration variables about zero. Moreover, fully linearized equations governing motions of beams such as C can be written which, for $r = 7, 8, 9$ read

$$m_C(-LF_{r-6}\dot{u}_4 + E_{r-6}\dot{u}_{14} - G_{r-6}\dot{u}_r) - EJL\big|_C q_r \lambda^4_{r-6} = 0 \qquad (20)$$

where, with $\phi_i (i = 1, 2, 3)$ as in Eq. (9)

$$E_{r-6} = \frac{1}{L}\int_0^L \phi_{r-6}\,\mathrm{d}x \qquad F_{r-6} = \frac{1}{L^2}\int_0^L \phi_{r-6}\,x\,\mathrm{d}x$$

$$G_{r-6} = \frac{1}{L}\int_0^L \phi^2_{r-6}\,\mathrm{d}x$$

Similar equations govern the motion of D (with u_{15}, u_{16}, u_5, u_{10}, u_{11}, and u_{12} replacing u_{13}, u_{14}, u_4, u_7, u_8, and u_9, respectively). In summary, the motion of the unconstrained system represented by

$$F_r + F_r^* = 0 \qquad (r = 1, \ldots, 21) \qquad (21)$$

are given for $r = 1, 2, 3, 7, 8$, and 9 by Eqs. (19) and (20).

Now, the robot and the satellite move such $\hat{B}$ and $\bar{C}$, $\hat{C}$ and $\bar{D}$, $\hat{D}$ and $\bar{E}$, and $\hat{E}$ and $\bar{P}$ coincide, and P is attached to E, i.e.,

$${}^N\boldsymbol{v}^{\bar{C}} = {}^N\boldsymbol{v}^{\hat{B}} \qquad {}^N\boldsymbol{v}^{\bar{D}} = {}^N\boldsymbol{v}^{\hat{C}} \qquad {}^N\boldsymbol{v}^{\bar{E}} = {}^N\boldsymbol{v}^{\hat{D}}$$

$${}^N\boldsymbol{v}^{\bar{P}} = {}^N\boldsymbol{v}^{\hat{E}} \qquad {}^N\omega^P = {}^N\omega^E \qquad (22)$$

Substitutions from Eqs. (17) into Eqs. (22) lead to the following equations:

$$u_{13} = u_1 - (\hat{b}_1 s_3 + \hat{b}_2 c_3)u_3 \qquad (23)$$

$$u_{14} = u_2 + (\hat{b}_1 c_3 - \hat{b}_2 s_3)u_3 \qquad (24)$$

$$u_{15} = u_{13} + u_4\left[-Ls_4 - y^C(L)c_4\right] - \dot{y}^C(L)s_4 \qquad (25)$$

$$u_{16} = u_{14} + u_4\left[Lc_4 - y^C(L)s_4\right] + \dot{y}^C(L)c_4 \qquad (26)$$

$$u_{17} = u_{15} - \left[Ls_5 + y^D(L)c_5\right]u_5 - \dot{y}^D(L)s_5 + (\bar{e}_1 s_6 + \bar{e}_2 c_6)u_6 \qquad (27)$$

$$u_{18} = u_{16} + u_5\left[Lc_4 - y^D(L)s_4\right] + \dot{y}^D(L)c_5 - (\bar{e}_1 c_6 - \bar{e}_2 s_6)u_6 \qquad (28)$$

$$u_{19} = u_{17} + (\bar{p}_1 s_{21} + \bar{p}_2 c_{21})u_6 - (\hat{e}_1 s_6 + \hat{e}_2 c_6)u_6 \qquad (29)$$

$$u_{20} = u_{18} - (\bar{p}_1 c_{21} - \bar{p}_2 s_{21})u_6 + (\hat{e}_1 c_6 - \hat{e}_2 s_6)u_6 \qquad (30)$$

$$u_{21} = u_6 \qquad (31)$$

These can be solved explicitly for $u_{13}, \ldots, u_{21}$ in terms of $u_1, \ldots, u_{12}$ and cast in the form of Eqs. (2) (so that the dependent generalized speeds have the highest indices). Equations governing the motion of the system are obtained by substitutions from Eqs. (21) and (23–31) in Eqs. (4) for $m = 9$.

Consider a robot and a satellite with the following properties: $m_B = 93{,}500$; $m_C = 140$; $m_D = m_E = 87$; $m_P = 14{,}500$ kg; $J_B = 9{,}454{,}000$; $J_E = 150$; $J_P = 423{,}000$ kg-m^2; $L = 6.5$; $\hat{b}_1 = -11$; $\hat{b}_2 = 1.6$; $\bar{e}_1 = -0.9$; $\bar{e}_2 = 0$; $\hat{e}_1 = 0.9$; $\hat{e}_2 = 0$, $\bar{p}_1 = -1.5$; $\bar{p}_2 = 0$ m; and $EJL|_C = EJL|_D = 2.10^7$ N-m^3. Suppose that initially $q_4 = \pi/2$ rad; $q_{13} = q_{15} = -11$; $q_{14} = 1.6$; $q_{16} = q_{18} = q_{20} = 8.1$; $q_{17} = -3.6$; $q_{19} = -1.2$ m; $u_1 = 1.1\exp(-6)$; $u_2 = -2.3\exp(-6)$ m/s; $u_3 = 2.3\exp(-7)$; $u_4 = 0.051$; $u_5 = 0.17$; $u_6 = 0.1$ rad/s; and $u_7 = -0.2$; $u_8 = -0.015$; $u_9 = -0.026$; $u_{10} = -0.68$; $u_{11} = -0.05$; $u_{12} = -0.087$. Then $y^C(L, t)$ and $y^D(L, t)$, the elastic deflections of $\hat{C}$ and $\hat{D}$, are as shown in Fig. 3 for a 0.2-s simulation with an integration step of 0.00002 s. A fast Fourier transform (FFT) performed on these results reveals the following eigenfrequencies: 14.0, 56.8, 217.3 (associated with the deflection of C) and 17.8, 72.6, and 275.3 Hz (associated with the deflection of D). The mass properties of B and P exceed those of C and D considerably. Hence, the indicated frequencies compare with the first three eigenfrequencies of two pin-pin beam configurations identical with

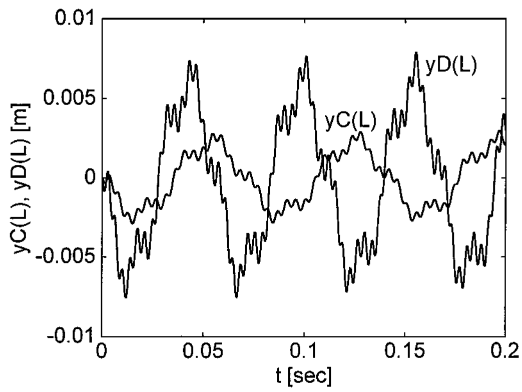

Fig. 3 Elastic deflections at $\hat{C}$ and $\hat{D}$.

those of C and D, namely, 14.0, 56.1, 126.3, and 17.8, 71.3, and 160.4 Hz. These are obtained by direct substitutions in the expression $1/2\pi(i\pi/L)^2(EJL/M)^{1/2}$ $(i = 1, 2, 3)$ (Ref. 9). Moreover, FFT of results of a similar run without P reveals the following eigenfrequencies: 15.0, 58.0, 218.0, and 20.0, 74.5, and 277.5 Hz. Modal analysis of the system performed with the aid of ANSYS,[10] with C and D comprising 15 beam elements each, gives rise to the following values: 14.8, 56.9, 126.7, and 19.8, 71.1, and 148.3.3 Hz. If agreement is to be obtained for higher frequencies, then additional modes should be used in Eqs. (18).

Conclusions

Kane's equations for constrained systems were used to generate motion equations of systems with elastic members. The ensuing elastic properties of the system were shown to be similar to those obtained by different methods. Now, suppose that a library of sets of motion equations of different kinds of elastic bodies, regarded as undergoing an unconstrained motion, is built. Then the generation of equations of motion of systems consisting of library-documented bodies reduces to the generation of the kinematical constraint equations and to the use of Kane's extended equation, consuming significantly less labor than required by conventional methods.

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